Reflection: its concepts and applications in geometry

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Received 20 December, 2004
Revised 12 January, 2005

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Abstract

This paper discusses the basic concepts of reflection and its related concepts in optics. It aims at providing examples on how to apply the principle of reflection in geometry. Explorations of the concepts involved via dynamic geometry software are also included.

Keywords: reflection, optics and dynamic geometry.

Reflection and it related concepts in optics

According to Kay (2001), the definition of reflection in a plane is as follows:

Definition: If a transformation f has the property that some fixed line l is the perpendicular bisector of the segment PP' for any point P in the plane and $P' = f(P)$, then f is a reflection with respect to l. The line l is called the line of reflection.

The point P' is called the image of reflection of P. The definition implies that P and P' are located at the opposite sides of l and they are equidistant from l. It is a basic property of reflection. A related concept in optics is the law of reflection, which states that the angle of incidence (the angle between the incoming ray and the normal to the reflecting surface) is equal to the angle of reflection (the angle between the outgoing ray and the normal), as illustrated in Figure 1.

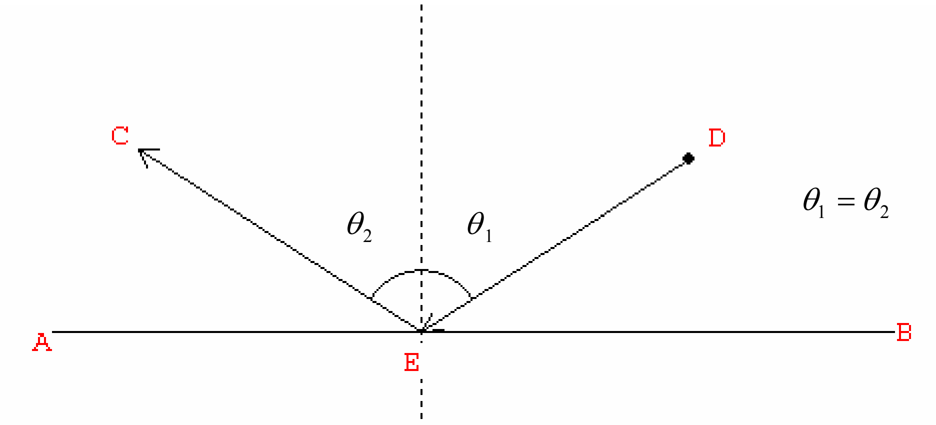

Figure 1

This phenomenon is due to the Fermat's principle, which states that light passing through a homogeneous medium (such as air) follows the shortest distance, in order to minimize energy (see Gay, D., 1998 and Hecht, E., 1998). The mathematical reason behind this law of optics can be explained in Figure 2, where D' denotes the image of reflection of D. Since the ray of light takes a path so as to minimize the total distance of travel $DE+EC$ $(= D'E+EC)$, therefore D', E and C must be collinear. Hence, we have $\angle CEA = \angle D'EB = \angle DEB$ and so $\theta_1 = 90^{\circ} - \angle DEB = 90^{\circ} - \angle CEA = \theta_2$.

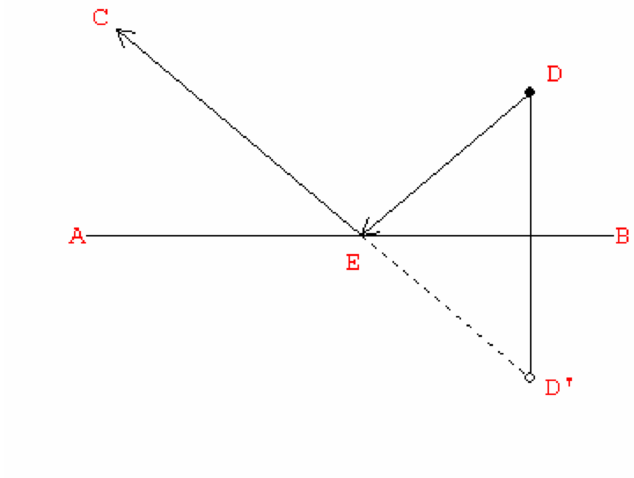

Figure 2

Using a dynamic geometry software, such as WinGeom or Sketchpad, this result can be further exemplified or explored by adding an arbitrary point *P* on *AB* and then measure the total distance traveled by the light ray from *A* to *E*. By dragging the point *P* along *AB*, we can see that *DP+PC (= D'P+PC)* is always greater than *DE+EC (= D'E+EC = D'C)*, provided *P* and *E* are distinct points, as shown in Figure 3. In fact, a careful study of the diagram reveals that it is simply a basic result in Euclidean Geometry, which states that the total length of any two sides of a triangle is always greater than the length of the third side.

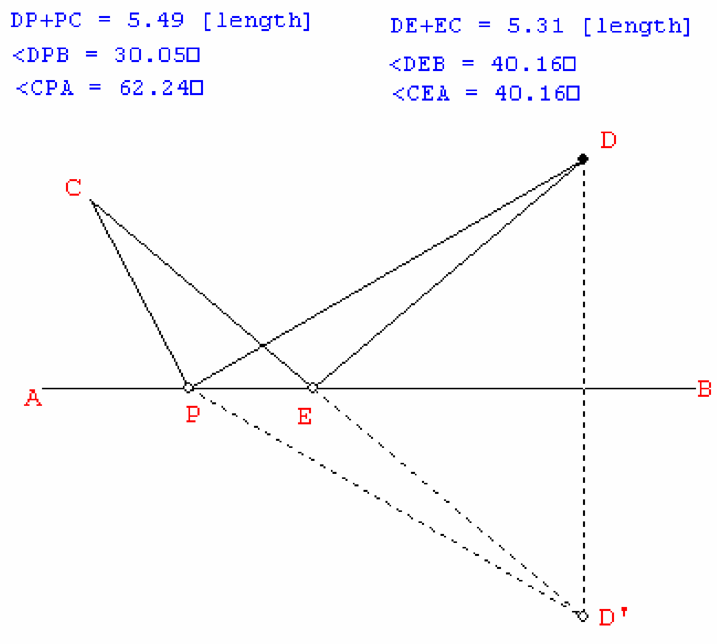

Figure 3

Some applications in geometry

We now illustrate how to apply the principle of reflection in geometry in the following examples.

Example 1 (The Cowboy Problem)

In Figure 4, a cowboy is located at C. Before riding on his horse to the campsite S, he wants to ride his horse to get some grass at the edge of the field and then go to the river for a drink. What is the shortest path from C to the edge of the grass field, to the river and then to S?

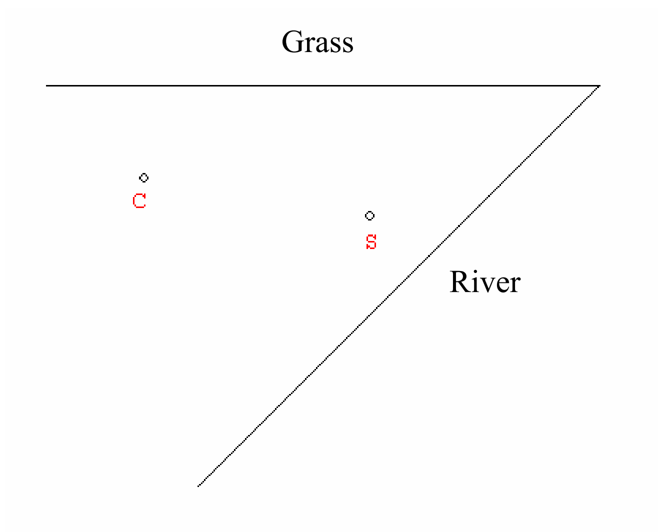

Figure 4

To start off, we have to look for the positions of the points P and Q, where the cowboy stops at the edge of the field and at the river, respectively. How can we locate these two points? First, we find the mirror images C' and S' by reflections. Next, we join C' and S' to obtain a line segment. The intersections between this line segment and the edges of the field and the river will fix the positions of P and Q. Thus, the shortest path required is $CPQS$, as shown in Figure 5.

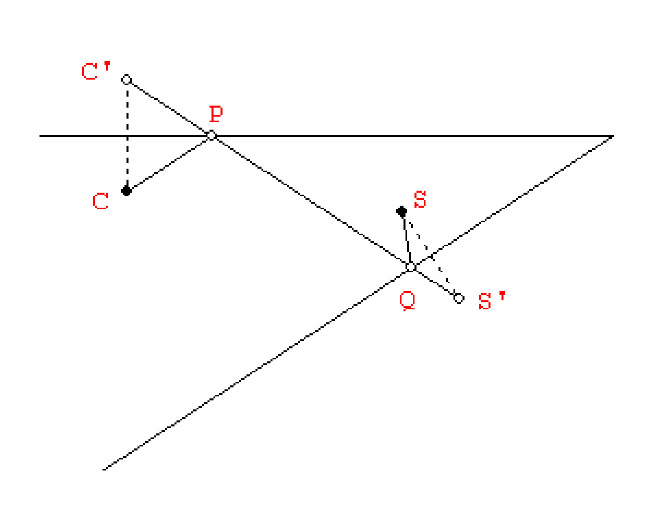

Figure 5

To prove its correctness, we first note that $CP + PQ + QS = C'P + PQ + QS'$, which means the total length of the path is equal to the length of $C'S'$. Now, suppose the cowboy stops at other arbitrary points on the edges, say P' and Q'. By reflections, the total distance traveled will be equal to $C'P' + P'Q' + Q'S'$, which is greater than $C'S'$ because $C'P'Q'S'$ is not a straight line, as illustrated in Figure 6. We can see that the built-in measurement tool of dynamic geometry software is quite convenient for illustration or self-exploration of the concepts involved.

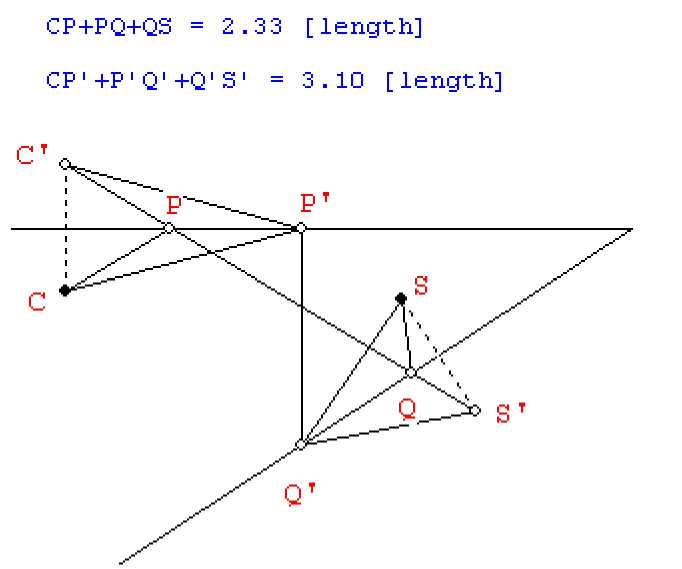

Figure 6

Example 2 (The Billiard Ball Problem)

In Figure 7, a billiard ball is placed at a point *D* in a triangular billiard table *ABC*. If the ball is projected to the side *AB*, and rebounds to *BC* and then to *CA*, what is the path of the ball if it can return to *D* after the three rebounds? Assume there is no energy lost when the ball hits the sides of the table.

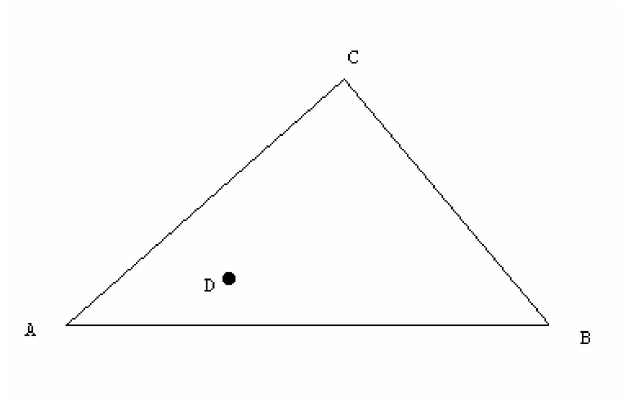

Figure 7

First, we have to look for the positions of the points *P*, *Q* and *R*, where the ball rebounds from *AB*, *BC* and *CA*, respectively. Applying reflections is a good start, so we find the mirror images D_1 and D_2 by reflections, with respect to *AB* and *CA*, respectively. Although we do not know where are the locations of *P*, *Q* and *R* yet, we can get some ideas by adding an arbitrary point *Q'* on *BC* and then draw the line segments $Q'D_1$ and $Q'D_2$, as shown in Figure 8. It is easy to see that the angle of incidence is equal to the angle of reflection at either E or F because $\angle DEA = \angle Q'EC$ and $\angle DFA = \angle Q'FB$. However, $\angle EQ'C$ may not be equal to $\angle FQ'B$. Again, a dynamic geometry software is a good tool for us to explore the exact position of *Q* by dragging *Q'* along *BC* and allow us to observe the measures of $\angle EQ'C$ and $\angle FQ'B$.

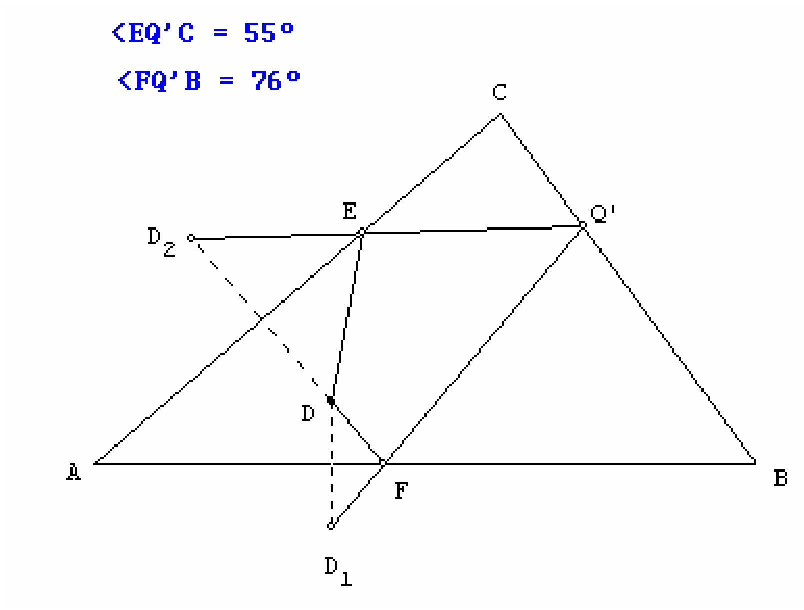

Figure 8

Figure 9 shows the desired position of *Q'* and so we can replace the labels *F*, *Q'* and *E* by *P*, *Q* and *R*, respectively, to obtain the path of the ball in Figure 10.

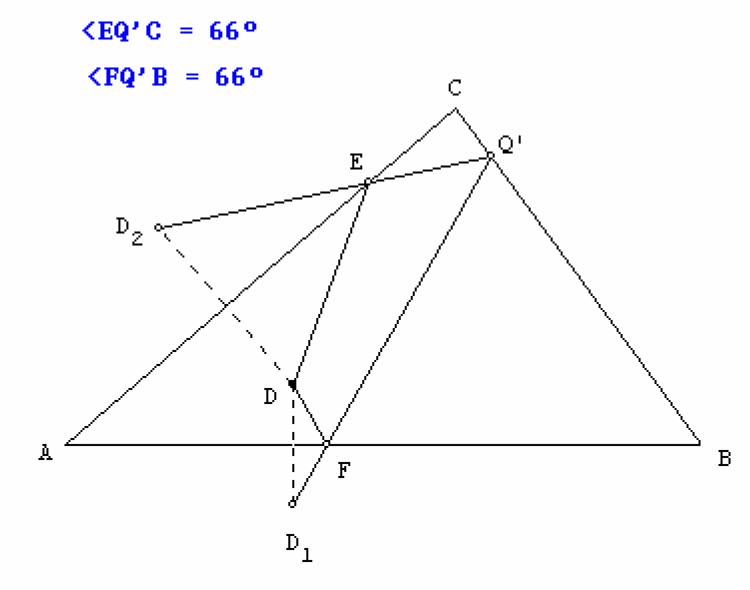

Figure 9

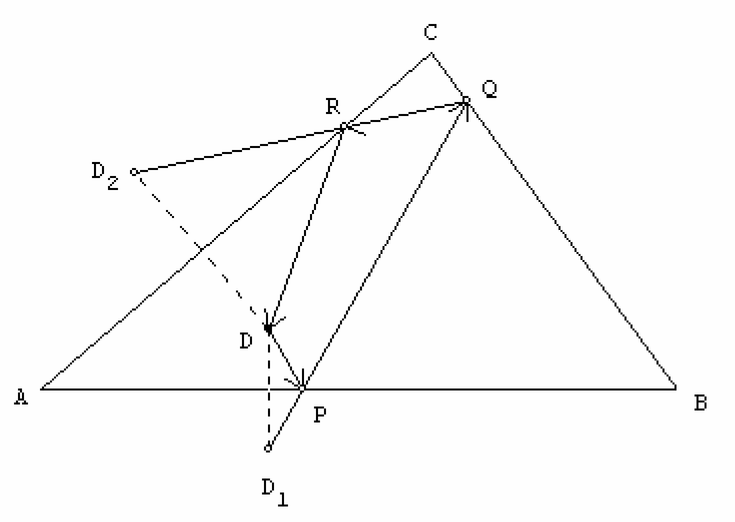

Figure 10

A further exploration by dynamic geometry software also reveals that the mirror image of D_1 with respect to BC, say D_3, must lie on the extension of D_2Q. It is because $\angle PQB = \angle D3QB = \angle RQC$, and so D_2, Q and D_3 are collinear, as shown in Figure 11. Hence, we know that a quick solution to this billiard ball problem is to find the mirror images D_1, D_2 and D_3 first, and draw the line segment D_2D_3 to locate the positions of R and Q, and then draw the line segment D_1Q to locate the position of P. Joining the points D, P, Q and R will give us the path required.

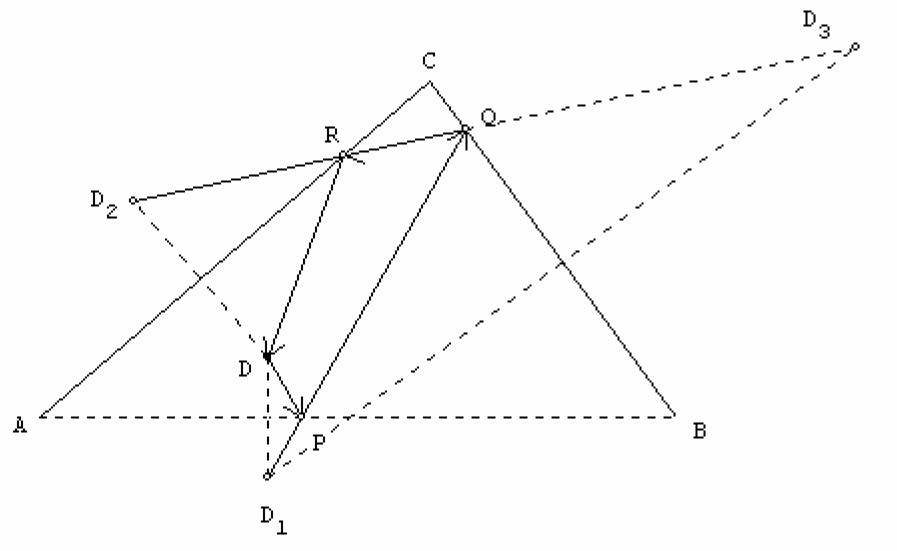

Figure 11

In fact, we can further deduce that the ball can return to its original position provided the line segment D_2D_3 intersects with the line segments BC and CA. However, we have to point out that this example is a slight deviation to the real-life situation in which the energy lost is normally inevitable and the angle of incidence need not be equal to the angle of reflection in such a case.

Final remarks

This paper discusses the basic concepts of reflection and its related concepts in optics. It aims at providing examples on how to apply the principle of reflection in geometry. The approach adopted to explain the ideas behind the method of solutions, via dynamic geometry software, is based on the author's own teaching experience. We hope the readers would find the discussions useful and be able to apply the same ideas to solve other related problems or use the examples in their own teaching, wherever appropriate.

Reference

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2. Gay, D. (1998). *Geometry by discovery*. New York: John Wiley & Sons.

3. Hecht, E. (1998). *Optics* (3rd edition). New York: Addison Wesley.